4. Να βρεθούν οι παράγωγοι των πραγματικών συναρτήσεων $y(x)$, $z(x)$ που ορίζονται με πεπλεγμένη μορφή, από το σύστημα:

$$\begin{cases} \sin(x+y) + z = 1 \\ \sin(x-y) + z = 2. \end{cases}$$

Λύση: Ας είναι,

$$\begin{cases} F_1(x, y(x), z(x)) = \sin(x+y) + z - 1 = 0 \\ F_2(x, y(x), z(x)) = \sin(x-y) + z - 2 = 0. \end{cases} \quad (1)$$

Από τον κανόνα της αλυσίδας παίρνω:

$$\begin{cases} dF_1(x, y(x), z(x)) = 0 \\ dF_2(x, y(x), z(x)) = 0 \end{cases} \Rightarrow \begin{cases} \frac{\partial F_1}{\partial x} \cdot dx + \frac{\partial F_1}{\partial y} \cdot dy + \frac{\partial F_1}{\partial z} \cdot dz = 0 \\ \frac{\partial F_2}{\partial x} \cdot dx + \frac{\partial F_2}{\partial y} \cdot dy + \frac{\partial F_2}{\partial z} \cdot dz = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\partial F_1}{\partial x} \cdot dx + \frac{\partial F_1}{\partial y} \cdot y'(x)dx + \frac{\partial F_1}{\partial z} \cdot z'(x)dx = 0 \\ \frac{\partial F_2}{\partial x} \cdot dx + \frac{\partial F_2}{\partial y} \cdot y'(x)dx + \frac{\partial F_2}{\partial z} \cdot z'(x)dx = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} \cdot y'(x) + \frac{\partial F_1}{\partial z} \cdot z'(x) = 0. \\ \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} \cdot y'(x) + \frac{\partial F_2}{\partial z} \cdot z'(x) = 0 \end{cases} \Rightarrow$$

$$\begin{cases} \dfrac{\partial F_1}{\partial y} \cdot y'(x) + \dfrac{\partial F_1}{\partial z} \cdot z'(x) = -\dfrac{\partial F_1}{\partial x} \\[2ex] \dfrac{\partial F_2}{\partial y} \cdot y'(x) + \dfrac{\partial F_2}{\partial z} \cdot z'(x) = -\dfrac{\partial F_2}{\partial x} \end{cases} \qquad (2\text{x}2 \text{ σύστημα})$$

Λύνω το παραπάνω 2x2 σύστημα ως προς $y'(x)$, $z'(x)$ με τη μέθοδο του Cramer και έχω:

$$y'(x) = \frac{\begin{vmatrix} -\dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial z} \\[2ex] -\dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial z} \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial F_1}{\partial y} & \dfrac{\partial F_1}{\partial z} \\[2ex] \dfrac{\partial F_2}{\partial y} & \dfrac{\partial F_2}{\partial z} \end{vmatrix}} , \quad z'(x) = \frac{\begin{vmatrix} \dfrac{\partial F_1}{\partial y} & -\dfrac{\partial F_1}{\partial x} \\[2ex] \dfrac{\partial F_2}{\partial y} & -\dfrac{\partial F_2}{\partial x} \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial F_1}{\partial y} & \dfrac{\partial F_1}{\partial z} \\[2ex] \dfrac{\partial F_2}{\partial y} & \dfrac{\partial F_2}{\partial z} \end{vmatrix}} \qquad (2)$$

(1) $\to$ (2) για τις ορισμένες F_1, F_2 έχω ότι:

$$y'(x) = \tan(x) \cdot \tan(y) , \quad z'(x) = \frac{\cos(2x)}{\cos x \cdot \cos y}$$